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NON-LINEAR PERTURBATIONS INDUCING A NATURAL PRESSURE GRADIENT IN A BOUNDARY LAYER ON A PLATE IN A TRANSONIC FLOW†

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The development of time-dependent non-linear perturbations in a laminar boundary layer on a plate in the case of transonic external flow is investigated. The study of the two-dimensional velocity field reduces to solving an integrodifferential equation for a function which depends on time and one spatial coordinate. The theory which is developed achieves a continuous transition from subsonic to supersonic flow since the above-mentioned governing equation contains the Burgers equation and the Benjamin–Ono equation, by means of which the evolution of perturbations outside the transonic range are described, as limiting cases.

THE GENERALIZATION of a triple-deck theory of the free interaction of a boundary layer [1–3] to the case of transonic velocities can lead to differing estimates of the scales of the perturbations depending on the role of unsteady effects. The problem with interaction in the transonic range, considered for the first time in [4], admits of the introduction of time in the external potential flow domain [5] without changing the estimates corresponding to stationary conditions. These estimates, as was shown in [6], are established, for example, from the similarity laws which occur in the theory [1–3]. Another mechanism for the wave propagation has been proposed in [7], where the scales of the variables introduced necessitate the retention of the time-dependent terms in the external domain as well as in the boundary domain.

Below, we describe an asymptotic analysis of non-linear perturbations, the amplitude of which exceeds the value assumed in [7]. The concept of self-induced pressure in the case of such amplitudes prescribes the normalization of the dependent and independent variables which is different from that in [4, 5, 7], with the flow acquiring a four-deck structure. Estimates of the quantities are introduced as combinations of the transonic parameter and powers of Reynolds number. A special limiting transition to Mach numbers of the order of unity reduces the asymptotic procedure under consideration to the previously proposed four-deck theory [8].

1. Let us study the longitudinal transonic gas flow around a flat plate under the assumption that a perturbation of the form

$$\frac{u^*}{U_\infty^*} = U_0 + \delta u_{1m} + \delta^2 u_{2m} + \ldots, \quad \frac{\upsilon^*}{U_\infty^*} = \delta^{5/2}\upsilon_{1m} + \delta^{7/2}\upsilon_{2m} + \ldots$$
$$\frac{\rho^*}{\rho_\infty^*} = R_0 + \delta\rho_{1m} + \delta^2\rho_{2m} + \ldots, \quad \frac{p^* - p_\infty^*}{\rho_\infty^* U_\infty^{*2}} = \delta^2 p_{1m} + \delta^3 p_{2m} + \ldots \tag{1.1}$$

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is introduced at a distance L^* from the leading edge into the bulk of the boundary layer.

Here, the x^* axis of the Cartesian system of coordinates x^*, y^* is directed along the velocity vector of the free stream with Mach number M_∞, t^* is the time, u^* and v^* are the components of the velocity vector, ρ^* is the density, p^* is the pressure and μ^* is the dynamic coefficient of viscosity. Dimensional quantities are indicated by asterisks, the subscript ∞ refers to the unperturbed state at infinity and the leading edge corresponds to $x^*=0$. The arguments of the functions with the subscripts $1m, 2m, \ldots$ are $T=\delta^{5/2}\mathrm{Re}^{1/2}U_\infty^* L^{*-1}t^*$, $X=\delta^{3/2}\mathrm{Re}^{1/2}L^{*-1}(x^*-L^*)$, $Y_m=\mathrm{Re}^{1/2}L^{*-1}y^*$. Reynolds numbers $\mathrm{Re}=\rho_\infty^* U_\infty^* L^*/\mu_\infty^\mu \to \infty$, are considered, and the amplitudes of the perturbations in (1.1) are specified in terms of a small parameter $\delta=(M_\infty^2-1)Q_\infty^{-1}$, where $Q_\infty=O(1)$. The length of the perturbed domain in which $X=O(1)$ is assumed to be small (compared to the scale L^*) which, on introducing the constraint $\mathrm{Re}^{-1/3}\ll\delta\ll 1$, enables one to regard the profiles of the longitudinal component of the velocity U_0 and the density R_0 of the initial steady-state solution in the boundary layer as depending solely on the vertical coordinate Y_m.

The above-mentioned constraint is a consequence of the stronger inequality

$$\mathrm{Re}^{-1/6} << \delta << 1 \tag{1.2}$$

which is assumed to be satisfied everywhere below. The meaning of (1.2) will be obvious from the subsequent discussion.

On substituting (1.1) into the Navier–Stokes equations, after integration we obtain

$$\begin{aligned}
&u_{1m}=A_1(T,X)\frac{dU_0}{dY_m}, \quad v_{1m}=-\frac{\partial A_1}{\partial X}U_0(Y_m), \quad \rho_{1m}=A_1\frac{dR_0}{dY_m}, \quad p_{1m}=p_{1m}(T,X)\\
&v_{2m}=-U_0(Y_m)\frac{\partial p_{1m}}{\partial X}\int_{Y_m}^{\infty}\left(\frac{1}{R_0U_0^2}-1\right)dY_m'-\frac{\partial A_1}{\partial T}-A_1\frac{\partial A_1}{\partial X}\frac{dU_0}{dY_m}-\frac{\partial A_2}{\partial X}U_0(Y_m)\\
&u_{2m}=-U_0(Y_m)p_{1m}-\int_{-\infty}^{X}\frac{\partial v_{2m}}{\partial Y_m}dX'\\
&\rho_{2m}=R_0p_{1m}+\frac{1}{U_0}\left[\frac{1}{2}A_1^2\left(U_0\frac{d^2R_0}{dY_m^2}-\frac{dU_0}{dY_m}\frac{dR_0}{dY_m}\right)-\frac{dR_0}{dY_m}\int_{-\infty}^{X}\left(\frac{\partial A_1}{\delta T}+v_{2m}\right)dX'\right]
\end{aligned} \tag{1.3}$$

The functions $A_1(T, X)$, $A_2(T, X)$, $p_{1m}(T, X)$ in (1.3) are arbitrary. The limiting behaviour of the unperturbed solution close to the wall, which we assume to be thermally insulated

$$U_0=\lambda_1 Y_m+\ldots, \quad R_0=r_0+\ldots, \quad (Y_m\to 0) \tag{1.4}$$

shows that, when $Y_m=O(\delta)$, the first two terms in the representation of both u^* and v^* from (1.1) become of the same order. Hence, in the lower part of the boundary layer, where the new vertical coordinate $Y_a=\delta^{-1}Y_m=\delta^{-1}\mathrm{Re}^{1/2}L^{*-1}y^*$ is of the order of unity, the perturbations become non-linear and the series (1.1) are transformed in the following manner

$$\frac{u^*}{U_\infty^*}=\delta u_{1a}+\ldots, \quad \frac{v^*}{U_\infty^*}=\delta^{7/2}v_{1a}+\ldots, \quad \frac{\rho^*}{\rho_\infty^*}=r_0+\ldots, \quad \frac{p^*-p_\infty^*}{\rho_\infty^* U_\infty^{*2}}=\delta^2 p_{1a}+\ldots \tag{1.5}$$

Functions with the subscript $1a$ depend on the arguments T, X and Y_a. As a result of representing their solution in the form of (1.5), the Navier–Stokes equations lead to

$$\frac{\partial u_{1a}}{\partial T}+u_{1a}\frac{\partial u_{1a}}{\partial X}+v_{1a}\frac{\partial u_{1a}}{\partial Y_a}=\frac{1}{r_0}\frac{\partial p_{1a}}{\partial X}, \quad \frac{\partial u_{1a}}{\partial X}+\frac{\partial v_{1a}}{\partial Y_a}=0, \quad \frac{\partial p_{1a}}{\partial Y_a}=0 \tag{1.6}$$

The non-viscous equations (1.6) imply the conservation of vorticity $\omega_{1a} = \partial u_{1a}/\partial Y_a$ along the trajectories of the gas particles. Assuming that the boundary layer is unperturbed at the initial instant of time, we find from this that

$$u_{1a} = \lambda_1 Y_{1a} + B_a(T,X), \quad \upsilon_{1a} = -Y_a \partial B_a(T,X)/\partial X + D_a(T,X)$$

The limits of the expansions (1.1) as $Y_m \to \infty$ rewritten in terms of the variable Y_a, which, when account is taken of the explicit expressions (1.3) and properties (1.4), have the form

$$\begin{gathered}\frac{u^*}{U_\infty^*} = \delta\lambda_1(Y_a + A_1) + \ldots, \quad \frac{\rho^*}{\rho_\infty^*} = r_0 + \ldots, \quad \frac{p^* - p_\infty^*}{\rho_\infty^* U_\infty^{*2}} = \delta^2 p_{1m} + \ldots \\ \frac{\upsilon^*}{U_\infty^*} = \delta^{7/2}\left(-\lambda_1 \frac{\partial A_1}{\partial X} Y_a - \frac{\partial A_1}{\partial T} - \lambda_1 A_1 \frac{\partial A_1}{\partial X} - \frac{1}{\lambda_1 r_0}\frac{\partial p_{1m}}{\partial X}\right) + \ldots\end{gathered} \tag{1.7}$$

must be identical to the limits of expansions (1.5) as $Y_a \to \infty$ in accordance with the matching principle. The condition which has been formulated, while uniquely defining the functions $B_a(T, X)$, $D_a(T, X)$, shows that $p_{1a} = p_{1m}$, and the principal terms with respect to δ of series (1.7) for u^* and υ^* yield the solution of Eqs (1.6).

Local roughness on the plate, the shape of which is specified by the equation

$$y^* = \delta \operatorname{Re}^{-1/2} L^* G_a(T,X) \tag{1.8}$$

can serve as a source of perturbations of the form being considered.

The impermeability condition

$$\upsilon_{1a} = \frac{\partial G_a}{\partial T} + u_{1a}\frac{\partial G_a}{\partial X}$$

on the surface $Y_a = G_a(T, X)$ of the deformed segment of the plate which is embedded in a non-linear non-viscous subdomain $Y_a = O(1)$ with the velocity field from (1.7) imposes the relationship

$$\frac{\partial(A_1 + G_a)}{\partial T} + \lambda_1(A_1 + G_a)\frac{\partial(A_1 + G_a)}{\partial X} = -\frac{1}{\lambda_1 r_0}\frac{\partial p_{1m}}{\partial X} \tag{1.9}$$

on the unknown functions $A_1(T, X)$, $p_{1m}(T, X)$.

In the case of the characteristic lengths and times introduced above, the thickness of the viscous sublayer is much smaller than the height of the roughness (1.8) if inequality (1.2) holds. In this sublayer, which is adjacent to the solid surface, it is appropriate to use the coordinate $N_1 = \delta^{9/4} \operatorname{Re}^{1/4}(Y_a - G_a) = O(1)$. The flow parameters are then written in the following form

$$\begin{gathered}\frac{u^*}{U_\infty^*} = \delta u_{1l} + \ldots, \quad \frac{\rho^*}{\rho_\infty^*} = r_0 + \ldots, \quad \frac{p^* - p_\infty^*}{\rho_\infty^* U_\infty^{*2}} = \delta^2 p_{1l} + \ldots \\ \frac{\upsilon^*}{U_\infty^*} = \delta^{7/2}\left(\frac{\partial G_a}{\partial T} + u_{1l}\frac{\partial G_a}{\partial X} + \delta^{-9/4}\operatorname{Re}^{-1/4}\upsilon_{1l}\right) + \ldots, \quad \frac{\mu^*}{\mu_\infty^*} = \mu_0 + \ldots\end{gathered} \tag{1.10}$$

Functions of the arguments T, X and N_i are marked with the subscript $1l$ and the dimensionless viscosity μ_0 was calculated using the temperature $T^*/T_\infty^* = 1/r_0$ of the unperturbed boundary layer when $Y_m = 0$.

The Navier–Stokes equations, when their solution is represented in the form of (1.10), yield

$$\frac{\partial u_{1l}}{\partial T}+u_{1l}\frac{\partial u_{1l}}{\partial X}+\upsilon_{1l}\frac{\partial u_{1l}}{\partial N_l}=-\frac{1}{r_0}\frac{\partial p_{1l}}{\partial X}+\frac{\mu_0}{r_0}\frac{\partial^2 u_{1l}}{\partial N_l^2} \tag{1.11}$$

$$\frac{\partial u_{1l}}{\partial X}+\frac{\partial \upsilon_{1l}}{\partial N_l}=0,\quad \frac{\partial p_{1l}}{\partial N_l}=0$$

To the no-slip conditions at the solid surface

$$u_{1l}=\upsilon_{1l}=0,\quad (N_l=0) \tag{1.12}$$

it is necessary to add the matching condition at the lower boundary of the non-viscous domain which implies that $p_{1l}=p_{1m}$ and

$$u_{1l}\to\lambda_1(A_1+G_a),\quad (N_l\to\infty) \tag{1.13}$$

Let us now turn our attention to the fact that the coordinates N_l and Y_a become of the same order when $\delta=\mathrm{Re}^{-1/9}$. In the latter case, the non-linear domain merges with the viscous sublayer adjacent to the wall and the scales of the perturbations are identical to those considered in [7].

2. Let us put $Y_m\to\infty$. The limiting properties $U_0(Y_m)\to 1$, $R_0(Y_m)\to 1$ then specify the behaviour of expressions (1.3) and also the asymptotic form of the solution (1.1) at the outer edge of the boundary layer

$$\frac{u^*}{U_\infty^*}\to 1-\delta^2 p_{1m}+\ldots,\quad \frac{\upsilon^*}{U_\infty^*}\to-\delta^{5/2}\frac{\partial A_1}{\partial X}+\ldots$$
$$\frac{\rho^*}{\rho_\infty^*}\to 1+\delta^2 p_{1m}+\ldots,\quad \frac{p^*-p_\infty^*}{\rho_\infty^* U_\infty^{*2}}\to\delta^2 p_{1m}+\ldots \tag{2.1}$$

If the thickness of the three flow subdomains which have been introduced above is smaller in order of magnitude than their length in the longitudinal direction the situation is the opposite in the external subdomain above the boundary layer: in the case of transonic flow, the scale of the vertical coordinate $y^*=\delta^{-2}\mathrm{Re}^{-1/2}L^*Y_u$, where the new variable $Y_u=O(1)$, exceeds the remaining previous scale $\delta^{-3/2}\mathrm{Re}^{-1/2}$ with respect to the coordinate x^*. When $Y_u\to 0$, the perturbations (2.1) induce in the external domain $Y_u\to O(1)$ a gas parameter field

$$\frac{u^*}{U_\infty^*}=1+\delta^2 u_{1u}+\delta^3 u_{2u}+\ldots,\quad \frac{\upsilon^*}{U_\infty^*}=\delta^{5/2}\upsilon_{1u}+\delta^{7/2}\upsilon_{2u}+\ldots$$
$$\frac{\rho^*}{\rho_\infty^*}=1+\delta^2\rho_{1u}+\delta^3\rho_{2u}+\ldots,\quad \frac{p^*-p_\infty^*}{\rho_\infty^* U_\infty^{*2}}=\delta^2 p_{1u}+\delta^3 p_{2u}+\ldots,\quad M_\infty^2=1+\delta Q_\infty \tag{2.2}$$

Functions with the subscript $1u, 2u, \ldots$ depend on the arguments T, X and Y_u. On introducing formulae (2.2) into the system of Navier–Stokes equations, we obtain

$$p_{1u}=\rho_{1u}=-u_{1u},\quad u_{1u}=\frac{\partial\varphi_{1u}}{\partial X},\quad \upsilon_{1u}=\frac{\partial\varphi_{1u}}{\partial Y_u} \tag{2.3}$$

In the next approximation, we have

$$\frac{\partial\rho_{1u}}{\partial T}+\frac{\partial u_{2u}}{\partial X}+\frac{\partial\rho_{2u}}{\partial X}+\frac{\partial\upsilon_{1u}}{\partial Y_u}=0,\quad \frac{\partial u_{1u}}{\partial T}+\frac{\partial u_{2u}}{\partial X}+\frac{\partial p_{2u}}{\partial X}=0 \tag{2.4}$$

$$\frac{\partial \upsilon_{1u}}{\partial T}+\frac{\partial \upsilon_{2u}}{\partial X}+\frac{\partial p_{2u}}{\partial Y_m}=0, \quad \frac{\partial p_{1u}}{\partial T}+\frac{\partial p_{2u}}{\partial X}=\frac{\partial \rho_{1u}}{\partial T}-Q_\infty\frac{\partial \rho_{1u}}{\partial X}+\frac{\partial p_{2u}}{\partial X}$$

Relationships (2.3) are supplemented by the equation for the potential

$$2\frac{\partial^2 \varphi_{1u}}{\partial T \partial X}+Q_\infty\frac{\partial^2 \varphi_{1u}}{\partial X^2}-\frac{\partial^2 \varphi_{1u}}{\partial Y_u^2}=0 \tag{2.5}$$

which is the condition for the system of equations to be solvable in the second approximation (2.4). The boundary conditions

$$\frac{\partial \varphi_{1u}(T,X,0)}{\partial X}=-p_m(T,X), \quad \frac{\partial \varphi_{1u}(T,X,0)}{\partial Y_u}=-\frac{\partial A_1(T,X)}{\partial X} \tag{2.6}$$

follow from the matching with (2.1).

The affine transformation

$$\begin{aligned} &T=b_t t, \quad X=b_x x, \quad Y_a=b_y y_a, \quad N_l=b_y n_l, \quad Y_u=b_y^* y_u \\ &u_{1a}=b_u u, \quad \upsilon_{1a}=b_\upsilon \upsilon, \quad u_{1l}=b_u \upsilon_l, \quad \upsilon_{1l}=b_\upsilon \upsilon_l, \quad p_{1m}=b_p p \\ &A_1=b_y A', \quad G_a=b_y G, \quad \varphi_{1u}=b_\varphi \varphi, \quad Q_\infty=b_q K_\infty \end{aligned} \tag{2.7}$$

eliminates the constants λ_1, r_0 and μ_0 from the subsequent treatment if one puts

$$\begin{aligned} &b_t=(2^2\lambda_1^{14}r_0^{-1}\mu_0^5)^{-1/9}, \quad b_x=(2\lambda_1^4 r_0\mu_0)^{-1/3}, \quad b_y=(2\lambda_1^7 r_0^4\mu_0^{-2})^{-1/6} \\ &b_u=(2^{-1}\lambda_1^2 r_0^{-4}\mu_0^2)^{1/9}, \quad b_\upsilon=(2\lambda_1^7 r_0^{-5}\mu_0^7)^{1/6}, \quad b_p=(2^{-2}\lambda_1^4 r_0\mu_0^4)^{1/6} \\ &b_y^*=(2^7\lambda_1^{13}r_0\mu_0^4)^{-1/9}, \quad b_\varphi=(2^5\lambda_1^8 r_0^2\mu_0^{-1})^{-1/9}, \quad b_q=(2^8\lambda_1^2 r_0^{-4}\mu_0^2)^{1/9} \end{aligned} \tag{2.8}$$

in (2.7).

Let us use the notation $A=A'+G$. Provided that there is no singularity in the solution of the problem

$$\frac{\partial u_l}{\partial t}+u_l\frac{\partial u_l}{\partial x}+\upsilon_l\frac{\partial u_l}{\partial n_l}=-\frac{\partial p}{\partial x}+\frac{\partial^2 u_l}{\partial n_l^2}, \quad \frac{\partial u_l}{\partial x}+\frac{\partial \upsilon_l}{\partial n_l}=0 \tag{2.9}$$

$$n_l=0: u_l=\upsilon_l=0, \quad n_l\to\infty: u_l\to A(t,x) \tag{2.10}$$

the viscous sublayer adjacent to the wall has no effect, in the approximation being considered, on the flow parameters in the upper subdomains. The closed system of non-viscous equations

$$\frac{\partial^2 \varphi}{\partial t \partial x}+K_\infty\frac{\partial^2 \varphi}{\partial x^2}-\frac{\partial^2 \varphi}{\partial y_u^2}=0 \tag{2.11}$$

$$\frac{\partial A}{\partial t}+A\frac{\partial A}{\partial x}=-\frac{\partial p}{\partial x} \tag{2.12}$$

$$\frac{\partial \varphi(t,x,0)}{\partial x}=-p(t,x), \quad \frac{\partial \varphi(t,x,0)}{\partial y_u}=-\frac{\partial A(t,x)}{\partial x}+\frac{\partial G(t,x)}{\partial x} \tag{2.13}$$

enables one, independently of (2.9) and (2.10), to find the functions $A(t, x)$, $p(t, x)$ and, thereby, to find the velocity field in the non-linear non-viscous subdomain

$$u_a=n_a+A, \quad \upsilon_a=-n_a\frac{\partial A}{\partial x}+\frac{\partial G}{\partial t}+u_a\frac{\partial G}{\partial x}, \quad n_a=y_a-G \tag{2.14}$$

as well as in the bulk of the boundary layer using the explicit expressions (1.3).

On making the substitution

$$x_* = x-(1+K_\infty)t, \quad t_* = x+(1-K_\infty)t$$

in (2.11), we obtain a wave equation for which the following representation holds [9]

$$\varphi(t_*,x_*,0) = -\frac{1}{\pi}\iint_{S_*}\frac{\partial\varphi(\eta_*,\xi_*,0)}{\partial y_u}\frac{d\eta_* d\xi_*}{\sqrt{(t_*-\eta_*)^2-(x_*-\xi_*)^2}}$$

Here S^* is the domain of integration $\eta_* < t_* - |x_* - \xi_*|$. On returning to the variables t and x and eliminating $p(t,x)$ using (2.13) and (2.12), we have

$$\frac{\partial A}{\partial t}+A\frac{\partial A}{\partial x} = -\frac{1}{\pi}\frac{\partial^2}{\partial x^2}\iint_S\left[\frac{\partial A(\eta,\xi)}{\partial\xi}-\frac{\partial G(\eta,\xi)}{\partial\xi}\right]\frac{d\eta d\xi}{\sqrt{(t-\eta)(x-\xi)-K_\infty(t-\eta)^2}} \tag{2.15}$$

where the integration sector S is established by the inequalities

$$\eta < t, \quad \xi < x - K_\infty(t-\eta) \tag{2.16}$$

When $K_\infty \to +\infty$, Eq. (2.15) becomes the Burgers equation

$$\frac{\partial A}{\partial t}+A\frac{\partial A}{\partial x} = \frac{1}{\sqrt{K_\infty}}\left[\frac{\partial^2 A}{\partial x^2}-\frac{\partial^2 G}{\partial x^2}\right] \tag{2.17}$$

Actually, the domain of integration (2.16) in this limit is enclosed between rays which make a small angle. Consequently, the integrand $A(\eta,\xi)$ (as well as $G(\eta,\xi)$) can be replaced by $A(t,\xi)$, and, after integrating with respect to the variable η, we then obtain (2.17) instead of (2.15).

The other limiting case when $K_\infty \to -\infty$ corresponds to integration within an obtuse angle close to π. The narrow domain of small $t-\eta$ makes the main contribution since, in the remaining part of the sector (2.16), double differentiation with respect to the variable x in Eq. (2.15) leads to a quantity of the order of $|K_\infty|^{-5/2}$. As above, let us replace the function $A(\eta,\xi)$ for small $t-\eta$ by $A(t,\xi)$, and then, after some simple reduction, we arrive at the Benjamin–Ono equation [10, 11]

$$\frac{\partial A}{\partial t}+A\frac{\partial A}{\partial x} = \frac{1}{\pi\sqrt{-K_\infty}}\int_{-\infty}^{\infty}\left[\frac{\partial A^2(t,\xi)}{\partial\xi^2}-\frac{\partial G^2(t,\xi)}{\partial\xi^2}\right]\frac{d\xi}{\xi-x} \tag{2.18}$$

Here, the integral is understood in the sense of a principal Cauchy value.

The integrodifferential equation (2.15) is a consequence of the interaction of a non-viscous boundary subdomain with an external potential flow through the bulk of the boundary layer, which plays a passive role. The non-trivial question concerning the existence of a regular solution of the Prandtl equations (2.9) with a pressure gradient specified using the function $A(t, x)$, found from (2.15) serves as an intrinsic criterion of the realizability of the four-deck structure of the perturbations which has been introduced above.

3. Let the surface of the plate be unperturbed ($G=0$). A non-linear motion which is induced by some method is then governed by Eq. (2.15) with a uniform right-hand side $\Pi(A)$ The equivalent way of writing out the operator $\Pi(A)$ in the form

$$\Pi(A) = \frac{1}{\pi}\frac{\partial^2}{\partial x^2}\int_0^\infty d\tau\int_0^\infty\frac{\partial A(t-\tau, x-K_\infty\tau-\nu)}{\partial\nu}\frac{d\nu}{\sqrt{\tau\nu}}$$

is convenient when the solution is sought in the form of a travelling wave $A = A(\zeta)$, $\zeta = x - ct$. When $K_\infty < c$, we have from (2.15)

$$(A-c)\frac{dA}{d\zeta} = \frac{1}{\sqrt{K_\infty - c}}\frac{d^2A}{d\zeta^2}$$

When $K_\infty < c$, we again arrive at the Benjamin–Ono equation.

A single soliton solution of Eq. (2.15)

$$A = \frac{4c}{1 + c^2(c - K_\infty)(x - ct)^2} \tag{3.1}$$

exists when $K_\infty < c < 0$. As can be seen from (3.1), the phase velocity of the soliton uniquely defines its amplitude $4c$. The "mass" of the soliton

$$\int_{-\infty}^{\infty} A dx = \frac{4\pi}{(c - K_\infty)^{1/2}} \tag{3.2}$$

is a function of the parameters occurring in (3.1). We note that, in the subsonic range, a quantity, analogous to (3.2), is independent of the phase velocity in the case of a single soliton perturbation. The integral characteristic (3.2) is important in the mechanism of the generation of solitons [12].

The periodic solution of Eq. (2.15)

$$A = \frac{2k^2}{c(c - K_\infty)}\left\{1 - \left[1 - \frac{k^2}{c^2(c - K_\infty)}\right]^{1/2} \cos[k(x - ct)]\right\}^{-1} \tag{3.3}$$

also assumes that $K_\infty < c < 0$.

Following [13], let us represent function (3.3) in the form of a chain of equidistant solitons

$$A = \sum_{n=-\infty}^{\infty} \frac{4s}{1 + s^2(c - K_\infty)(x - ct - 2n\pi k^{-1})^2} \tag{3.4}$$

where

$$s = c\frac{\theta}{\operatorname{arcth}\theta}, \quad \theta = \frac{k}{c(c - K_\infty)^{1/2}}, \quad c < 0 \tag{3.5}$$

Each term in (3.4) tends to the solution of a type of solitary wave (3.1) in the limit when $s \to c$ which is attained, according to (3.5), when $\theta \to 0$. Since $|s| < |c|$ always, in the general case the individual elements of the chain (3.4) are not solutions of Eq. (2.15) but only satisfy this equation in the sum as a result of the non-linear interaction.

Let α be a parameter. Equation (2.15) is invariant under the transformation

$$x \to x - \alpha t; \quad A \to A + \alpha, \quad K_\infty \to K_\infty - \alpha \tag{3.6}$$

and hence, along with (3.3), the function

$$A = c + \frac{k}{(c - K_\infty)^{1/2}}\left[\frac{1 + \sigma^2}{1 - \sigma^2} - 2F\right], \quad F = \frac{1 - \sigma^2}{1 + \sigma^2 - 2\sigma\cos[k(x - ct)]} \tag{3.7}$$

is also its solution. The four parameters appearing in (3.7) are governed by the constraints

$0<\sigma<1$, $c>K_\infty$, $k>0$. We note that, unlike (3.3), the phase velocity c in (3.7) can have any sign. In the case when

$$\sigma=\left(\frac{1-\Lambda}{1+\Lambda}\right)^{1/2}, \quad \Lambda=\frac{k}{|c|(c-K_\infty)^{1/2}}, \quad c<0$$

we return to (3.3). The identity

$$F=1+2\sum_{n=1}^{\infty}\sigma^n\cos[nk(x-ct)]$$

implies that the amplitudes of the harmonics in the Fourier expansion of the function (3.7) are not series in the parameter σ but are single terms in powers of n.

4. We will now consider the question of the existence of a regular solution in a boundary sublayer $n_l=O(1)$ when the periodic function (3.7) is chosen as the external boundary condition when $n_l\to\infty$. Equation (2.15) is invariant under the transformation

$$t\to\beta^{5/3}t, \quad x\to\beta x, \quad A\to\beta^{-2/3}A, \quad K_\infty\to\beta^{-2/3}K_\infty \tag{4.1}$$

which depends on the parameter β. The family of solutions (3.7) is mapped on to itself by the transformation (4.1) since its action on a function from the given family is equivalent to the substitution $k\to\beta k$, $c\to\beta^{2/3}c$, $K_\infty\to\beta^{-2/3}K_\infty$.

We will first consider a stationary 2π-periodic function

$$A=\frac{1}{(-K_\infty)^{1/2}}\left[-1+2\sum_{n=1}^{\infty}\left(\sigma^{2n}-2\sigma^n\cos nx\right)\right] \tag{4.2}$$

from the family (3.7). We will seek a stationary 2π-periodic solution of the system of Prandtl equations (2.9) with a pressure gradient $dp/dx=-\frac{1}{2}dA^2/dx$ specified in terms of A from (4.2) and the boundary conditions $u_l\to A$ $(n_l\to\infty)$; $u_l=c_0$, $\upsilon_l=0$ $(n_l=0)$. Here, we introduce the velocity c_0 of the wall along a tangent to itself as the additional parameter of the problem. It will follow from the subsequent account that the solution sought does not exist for all c_0.

Let us put

$$c_0=\frac{1}{(-K_\infty)^{1/2}}\left(-S_0-\sum_{n=1}^{\infty}\sigma^nS_n\right) \tag{4.3}$$

The system of Prandtl equations for the boundary layer undergoes the transformation

$$t\to\gamma_1\gamma_2^{-1}t, \quad x\to\gamma_1 x, \quad y_l\to(\gamma_1\gamma_2^{-1})^{1/2}y_l \tag{4.4}$$

$$u_l\to\gamma_2u_l, \quad \upsilon_l\to(\gamma_1^{-1}\gamma_2)^{1/2}\upsilon_l, \quad p\to\gamma_2^2p$$

which depends on the two parameters γ_1 and γ_2. Let us take $\gamma_1=-1$ and $\gamma_2=c_0<0$. Then, the application of (4.4) leaves system (2.9) unchanged, and the boundary conditions take the form

$$u_l=1, \quad \upsilon_l=0 \quad (n_l=0); \quad u_l\to u_\infty \quad (n_l\to\infty) \tag{4.5}$$

where $u_\infty=Ac_0^{-1}$. From (4.2) and (4.3) we have

$$u_\infty=\frac{1}{S_0}\left[1-\sigma\left(\frac{S_1}{S_0}-4\cos x\right)-\sigma^2\left(2+\frac{S_2}{S_0}-\frac{S_1^2}{S_0}+4\frac{S_1}{S_0}\cos x-4\cos 2x\right)+\ldots\right] \tag{4.6}$$

The conditions [14]

$$\int_0^{2\pi} Q_1 dx = 0, \quad \int_0^{2\pi} (u_\infty^3 - u_\infty)dx = O(\sigma^4) \tag{4.7}$$

for a stationary 2π-periodic solution of the system of Prandtl equations to exist were invoked in the case of the boundary-value problem (4.5) with a function u_∞ which can be represented in the form

$$u_\infty^2 = 1 + \sum_{n=1}^{\infty} \sigma^2 Q_n(x), \quad Q_n(0) = Q_n(2\pi)$$

The second condition from (4.7) holds if the Fourier series for the quantity $Q_2(x)$ only contains the first harmonic. The expansion (4.6) complies with this requirement.

Let us assume $S_0 = 1$. It then follows from (4.7) that $S_1 = 0$ and $S_2 = 10$. For brevity we will confine ourselves to the first terms of series (4.3) since the procedure for calculating the subsequent terms in the expansion with respect to σ is analogous to that given in [14].

The derivation of Eq. (2.15) and the formulation of problem (2.9), (2.10) corresponds to a fixed wall. Using (3.6), we change to a coordinate system where the wall is fixed. Returning from u_∞ to A, we conclude that, in the case of an external boundary condition of the form

$$A = -c_0 + \frac{1}{(-c_0 - K_\infty)^{1/2}}\left\{-1 + 2\sum_{n=1}^{\infty} [\sigma^{2n} - 2\sigma^n \cos[n(x + c_0 t)]]\right\} \tag{4.8}$$

a 2π-periodic solution exists in each sublayer if the relationship

$$c_0 = -\frac{1}{(-c_0 - K_\infty)^{1/2}}\left(1 + \sum_{n=1}^{\infty} \sigma^n S_n\right) \tag{4.9}$$

which implicitly determines the phase velocity in (4.8), is satisfied.

We will now extend the result to any period using (4.1), where it is necessary to put $\beta = k$. Then, instead of (4.8), we return to the function (3.7), in which the phase velocity $c = k^{2/3}|c_0|$, as the external boundary condition. As far as the viscous sublayer is concerned, in the steady-state solution of the Prandtl equations for A specified using (4.2), it is necessary to make the substitution $u_l \to u_l + |c_0|$, $x_l \to x_l - |c_0| t$ with c_0 from (4.9), having made use of their invariance with respect to (3.6), and then to carry out the transformation (4.4) with $\gamma_1 = k$, $\gamma_2 = k^{-2/3}$. As a result of this substitution, the new solution of system (2.9) now satisfies conditions (2.10) with the function A from (3.7). We note that, in (4.9), the change in K_∞ on changing to a moving system of coordinates according to (3.6) has been taken into account.

Hence, a solution of the Prandtl equations in the viscous boundary sublayer with no-slip conditions at the fixed wall and a pressure gradient $\Pi(A)$, specified using the function A from (3.7), exists if the following relationship between the four parameters in (3.7) holds

$$k = c(c - K_\infty)^{2/3}(1 - 10\sigma^2 + \ldots) \tag{4.10}$$

For $\sigma = 0$, expression (4.10) is a dispersion relationship in the linear theory of the stability of a boundary layer in the case of transonic external flow velocities which agrees, apart from the scale factors (2.8), with that obtained in [15].

5. The parameter K_∞ may be eliminated from the Burgers equation (2.17) and the Benjamin–Ono equation (2.18), which hold in the case when $|K_\infty| \to \infty$, using an affine transformation, the choice of which is not unique. However, the transformation, which eliminates the parameter K_∞ from (2.17) and (2.18) with the additional requirement of the invariance of problem (2.9), (2.10) as well as relationships (2.14) with respect to it, is uniquely defined

$$
\begin{aligned}
&t \to |K_\infty|^{-1/4} t, \quad x \to |K_\infty|^{-3/8} x, \quad y_a \to |K_\infty|^{-1/8} y_a \\
&u_a \to |K_\infty|^{-1/8} u_a, \quad \upsilon_a \to |K_\infty|^{1/8} u_a, \quad p \to |K_\infty|^{-1/4} p \\
&A \to |K_\infty|^{-1/8} A, \quad G \to |K_\infty|^{-1/8} G
\end{aligned} \tag{5.1}
$$

The coefficients of the extension of the functions with the subscripts a and l are identical. Let $|K_\infty| = O(\delta^{-1})$. It follows from $M_\infty^2 = 1 + \delta Q_\infty$ and (2.7), (2.8) that

$$
|K_\infty| = \delta^{-1} 2^{-8/9} \lambda_1^{-2/9} r_0^{4/9} \mu_0^{-2/9} |M_\infty^2 - 1| \tag{5.2}
$$

Superposition of the two transformations (5.1) and (2.7), taking account of (5.2), yields

$$
\begin{aligned}
t^* &= \delta^{-9/4} \mathrm{Re}^{-1/2} L^* U_\infty^{*-1} \lambda_1^{-3/2} \mu_0^{-1/2} |M_\infty^2 - 1|^{-1/4} t^0 \\
x^* &= \delta^{-9/8} \mathrm{Re}^{-1/2} L^* \lambda_1^{-5/4} r_0^{-1/2} \mu_0^{-1/4} |M_\infty^2 - 1|^{-3/8} x^0 \\
y^* &= \delta^{9/8} \mathrm{Re}^{-1/2} L^* \lambda_1^{-3/4} r_0^{-1/2} \mu_0^{-1/4} |M_\infty^2 - 1|^{-1/8} y^0 \\
u^* &= \delta^{9/4} U_\infty^* \lambda_1^{1/4} r_0^{-1/2} \mu_0^{1/4} |M_\infty^2 - 1|^{-1/8} u^0 \\
\upsilon^* &= \delta^{27/8} U_\infty^* \lambda_1^{3/4} r_0^{-1/2} \mu_0^{3/4} |M_\infty^2 - 1|^{1/8} \upsilon^0 \\
p^* &= p_\infty^* + \delta^{9/4} \rho_\infty^* U_\infty^{*2} \lambda_1^{1/2} \mu_0^{1/2} |M_\infty^2 - 1|^{-1/4} p^0
\end{aligned} \tag{5.3}
$$

Dimensionless variables in the special system of units defined by (5.3) are indicated by the zero superscript and, as previously, dimensional quantities are indicated with an asterisk.

It is necessary to put $\Delta = \delta^{9/8}$ in (5.3). In terms of the small parameter Δ, we then obtain complete agreement with the normalization of the quantities in the four-layer theory [8]. In the case when $\Delta = \mathrm{Re}^{-1/8}$, we return to the classical version [1–3] of the theory of free interaction.

The order-of-magnitude relationship $|M_\infty^2 - 1| = O(\delta)$ in (5.3) makes the choice of the scales in (1.1), (1.5), (1.10), and (2.2) obvious. The continuous transition over the critical Mach number in the asymptotic equations obtained as a consequence of the estimates specified by (5.3), together with (5.2), subject to the condition $K_\infty = O(1)$, is associated with the non-linear nature of the flow.

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